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A Collocation Method for Second Order Boundary Value Problems

Ojobor, S.A

Department of Mathematics, Delta State University, P.M.B. 1 Abraka, Nigeria

Abstract
This paper is an extension of Mamadu and Ojobor (2017) were the efficiency of the collocation method was considered based on the type of basis function in developing the scheme. Here, we investigate the convergence of the method as applied to second order boundary value problems (BVPs) at the various collocation points: Gauss-Lobatto (G-L), Gauss-Chebychev (G-C) and Gauss-Radau (G – R) collocation points. Also, the class of Chebychev polynomials of the first kind have been adopted as basis function. We have employed Maple 18 software in our analysis and computations.
Keywords: Collocation method, basis functions, boundary value problems, collocation points, Chebychev polynomials

1. Introduction

Let the generalized form of a differential equation be given as

$$L[y(x)] = g(x),\ \varphi[y(a_1)] = a,\ \ \tau[y(a_2)] = b, \qquad (1.0)$$

where α, φ and τ are considered as differential operators.

Differential equation are often applied in the construction and development of most mathematical models such as predictive control in AP monitor (Hedengen et. al., 2014), temperature distribution in cylindrical conductor (Fortini et. al., 2008), dynamic optimization (cizinar et. al., 2015), etc. Modeling is the bridge between the subject and real-life situations for students realization. Differential equations model real-life situations, and provide the real-life answers with the help of computer calculations and graphics.

Investigation into methods for solving these problems has been on the increase in recent years. Obviously, many methods (analytical or numerical methods) have been developed and implemented by many researchers. Of these methods, the numerical methods seem to be more popular than their analytic counterpart due to their adequate approximation of the analytic solution in a rapid converging series. Popular numerical methods include; the Tau method (Adeniyi, 2004), orthogonal collocation method (Mamadu and Ojobor, 2017), Tau-Collocation method (Mamadu and Njoseh, 2016), Variation iteration decomposition method (Ojobor and Mamadu, 2017), Elzaki transform method (Mamadu and Njoseh, 2017), Power series approximation method (Njoseh and Mamadu, 2016a), Modified power series approximation method (Njoseh and Mamadu, 2017), etc.

However, the collocation method remains one of the best numerical method due to its level of simplicity and accuracy. Moreover, the efficiency of the method is dependent on the class of basis function and the collocation point adopted in constructing the scheme. There exist different types of basis functions that can be adopted to construct the scheme such as; canonical polynomials, Chebychev polynomials, Bernoulli polynomials, Lagrange polynomials (Fox and Pascal, 1968; Lanczos, 1938). And, the different collocations that can be adopted include; the equally spaced points; Gauss-lobotto points, Gauss-Chebychev points, Gauss-Radau point, etc. These points improves better than one another in terms of convergence.

This present study is an extension of Mamadu and Ojobor (2017) were the efficiency of the method was dependent on the type of basis function employed in constructing the scheme. Here, we investigate the convergence of the method as applied to the differential equation (1) at the various collocation points: Gauss-Lobatto (G-L), Gauss-Chebychev (G-C) and Gauss-Radau collocation points. We have also adopted the class of Chebychev polynomials of the first kind as our basis function in this study.

2. Chebychev polynomial of the first Kind

The Chebychev polynomial of the first kind is defined as (Njoseh and Mamadu, 2016b)

$$T_n(x) = cos(ncos^{-1}x) = \sum_{r=0}^{n} C_r^{(n)} x^r, \; -1 \leq x \leq 1, \quad (1.1)$$

with

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad (1.2)$$

satisfying the initial conditions $T_0(x) = 1$ and $T_1(x) = x$.

Now, if $-1 \leq x \leq 1$ mapped objectively to $a_1 \leq x \leq a_2$, then equation (1.2) becomes

$$T_{n+1}^*(x) = 2xT_n^*(x) - T_{n-1}^*(x), \quad (1.3)$$

satisfying the initial conditions $T_0^*(x) = 1$ and $T_1^*(x) = \frac{2x-a_1-a_2}{a_2-a_1}$.

Equation (1.3) is called the nth degree shifted Chebychev polynomials.

The Maple 18 execution code for generating the first kind Chebychev polynomials is given below:

```
> restart :
> N := ?
> ChebyshevT(n, x)
        (2)
> T[n + 1] := simplify((2), 'ChebyshevT') ,
```

Thus, the first ten Chebychev polynomials of the first kind as given as follows:

> $T_0 := 1$

> $T_1 := x$

> $T_2 := 2x^2 - 1$

> $T_3 := 4x^3 - 3x$

> $T_4 := 8x^4 - 8x^2 + 1$

> $T_5 := 16x^5 - 20x^3 + 5x$

> $T_6 := 32x^6 - 48x^4 + 18x^2 - 1$

> $T_7 := 64x^7 - 112x^5 + 56x^3 - 7x$

> $T_8 := 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$

> $T_9 := 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$

> $T_{10} := 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$

2.1 Colloaction Points

In this section, we outline our collocation points relevant to this research. Thus, our collocation points shall be:

2.1.1 Gauss-Radau (G – R).

Here, we collocate equation (1.5) at

2.1.2 Gauss-Lobatta (G – L).

Here, we collocate equation (1.5) at

2.1.3 Gauss-Chebychev (G – C).

Here, we collocate equation (1.5) at

3. Mathematical Formulation of the problem

Let define the approximate solution as

$$y(x) = \sum_{r=0}^{N} a_i T_i(x) \tag{1.4}$$

where $a_i, i = 0(1)N$, are unknown coefficients to be determined.

The major requirement in collocation method is that equation (1.0) must satisfy the collocation points in the interval $a_1 < x < a_2$.

The method is now given in detail.

Substituting equation (1.4) into equation (1.0), we have

$$L[y(x_i)] = g(x_i), \quad x_i \in (a_1, a_2), \qquad i = 1(2)(N-1) \tag{1.5}$$

$$\varphi[y(x_0)] = a, \quad \tau[y(x_N)] = b, \tag{1.6}$$

We then collocate equation (1.5) at the different collocation points: G - L, G - C and G – R. Note that i must equal the number of $a_i, i = 0(1)N$ in the approximate solution to overcome over-determination (Mamadu and Njoseh, 2016). Thus, we obtain (N-1) set of equations in (N-1) unknowns. Two more equations come from using the boundary conditions

$$\sum_{r=0}^{N} a_i T_i(a_1) = a, \quad \sum_{r=0}^{N} a_i T_i(a_2) = b. \tag{1.7}$$

We thus solve the resulting (N+1) equations for $a_i, i = 0(1)N$ using the Gaussian elimination method.

Readers are referred to Mamadu and Ojobor (2017) for the error analysis of the method.

3.1 Maple 18 Execution code for the Problem

We collocate (1.5) with the Maple 18 executive code as given below:

```
restart :

t := [ ] :# Please insert the collocation points ( be it G-L, G-R or G-C) for any value of N
for i from 1 to N do

x := t[i] :
A[i] := evalf(sort(simplify(L[y(x)])))
end do:

for i from 1 to N do

printf("\n Collocating at t=%f, we have: ", t[i]);

A[i];
end do;
```

Thus, solving the equations obtained after collocation and those in (1.7) with the help of Maple 18 code below

```
values := solve({A[1], A[2], A[3], A[4], A[5], A[6] ... A[N], B[1], B[2]}, {a[1], a[2], a[3],
    a[4], a[5], a[6] ...a[N + 2], }) :

assign(values) :
```

we obtain the unknown constants $a_i, i = 0(1)N$.

Implementing the command

```
printf("\n subtituting our values of a_1 = %f, a_2 = %f, a_3 = %f ,a_4=%f,a_5=%f , a_6=%f,..
  ., a_N+1=%f; we obtain the table below \n \n", a[1], a[2], a[3], a[4], a[5], a[6],...a[N
  + 2])
```

We obtain our approximate solution.

4. Numerical Experiment

We solve some second order boundary value problems (BVPs) using the collocation method with Chebychev basis function at G – L, G – C and G – R collocation points.

Example 4.1

Consider the second order BVP

$$(1 + x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0, x \in [-1,1], \quad (1.8)$$

$$y(-1) = y(1) = \frac{1}{2} \quad (1.9)$$

The Exact solution is given as $(x) = \frac{1}{1+x^2}$.

If N=4, the Maple 18 execution at the various collocation points for (1.8), we obtain the plot:

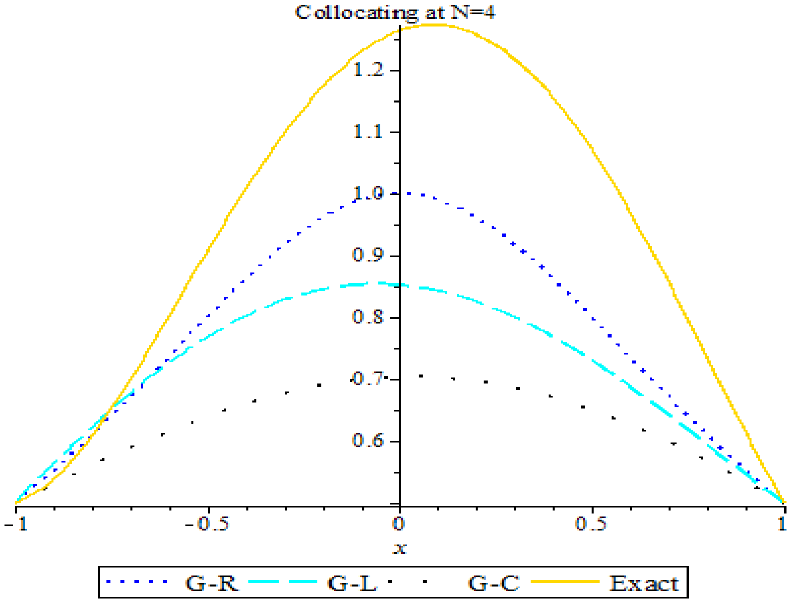

Figure 1. Comparison of the exact and approximate solutions at various collocation points for Example 4.1

Similarly at N=8, we obtain

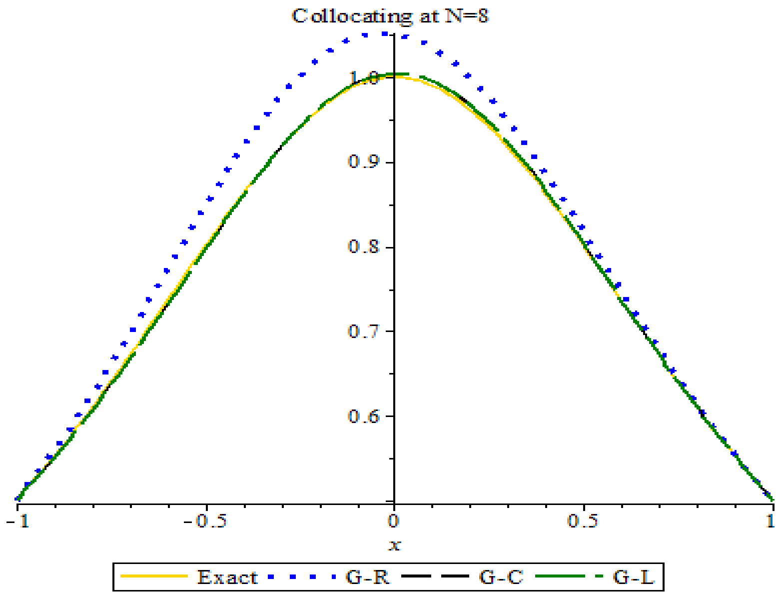

Figure 2. Comparison of the exact and approximate solutions at various collocation points for Example 4.1

Example 4.2.

Solve the linear initial value problem in second order ordinary differential equation

$$y''(x) + y(x) = 0, y(0) = 1, y'(1) = 3. \quad (2.0)$$

The exact solution is given as $y(x) = cosx - \frac{sinx\,(cos\,1-3)}{sin\,1}$.

If we let N=8, the Maple 18 execution at the various collocation points for (2.0) ,we obtain the plot:

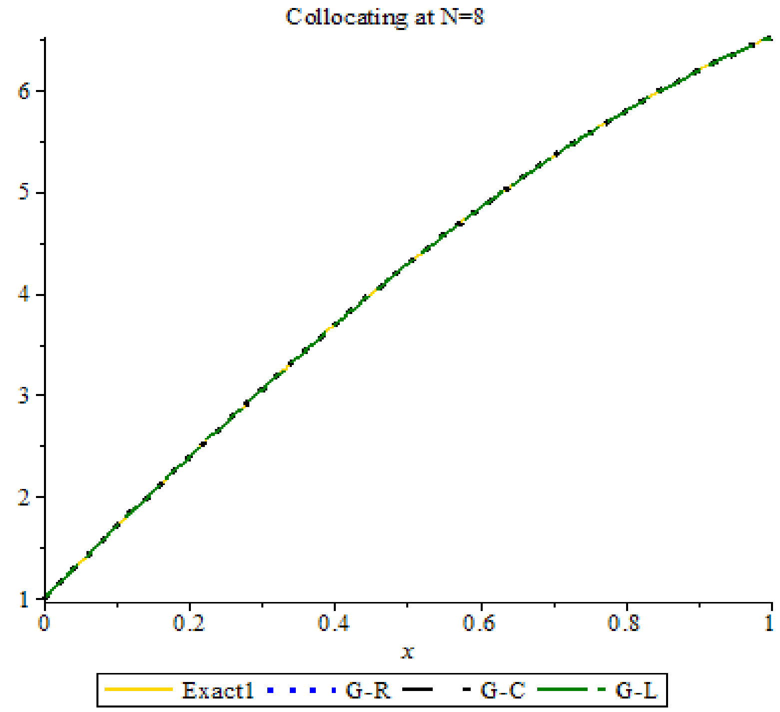

Figure 3. Comparison of the exact and approximate solutions at various collocation points for Example 4.2

5. Discussion of Results

We have demonstrated the collocation method at different points as in Example 4.1.and 4.2. In Example 4.1, at N=4, the G – R collocation point converges better than the others with a maximum error of $3.0000E - 10$ as shown in figure 1. Also, when N=8, the approximate solution blow up for G – R collocation point, while G-R converges better than the G-C collocation point with a maximum error of $3.0000E - 10$ as shown in figure 2. Similarly, at N=8, there is an absolute convergent at all collocation points considered for Example 4.2 as shown in figure 3.

6. Conclusion

We have vividly implemented the collocation method for solving second order boundary value problems at various collocation points, namely, G – R, G – L and G – C. From the resulting numerical evidences with the aid of Maple 18 software, we conclude that;

i. whenever N is small, the G – R collocation point is more preferable; and

ii. when N is Large, either G – L or G – C collocation points is more preferable in ensuring a rapidly convergent series solution.

Further Research

Future researchers should be enable to investigate how small N will be for the G – R collocation point to be preferred, or how large N will be for G – L or G – C collocation points to be preferable in ensuring a rapidly convergent series solution.

References

Hedengen, J.D., Asgharzadeh, S.R., powee, K.M., and Edgar, T.F., "Nonlniear modeling, estimation and predictive control in Apmonitor," Computers and Chemical Engineering, **70:**133-148, 2014.

Fortini, M.A., Stamoulis, M.N., Ferreira, F.M., Claubia, P., Costa, A.L. and Silver, C.A.M., "Application of the orthogonal collocation method to determination of temperature distribution in cylindrical conductor," Annals of Nuclear Energy, **35(9):**1681-1685, 2008.

Cizinar, M., Salhi, D., Fikar, M. and Latifi, M.A., "A matlab package for orthogonal coolcations on finite elements in dynamic optimization," 15th Int. Conference Process Control, Slokavia, 2015.

Adeniyi, R.B., "The Tau method in the numerical of integral equations," Abacus, **10:**261-274, 2004.

Mamadu, E.J. and Ojobor, S.A., "An orthogonal collocation approximation method for the analytic solution of Fredholm integral equations," Trans. Of NAMP, **3:**11-16, 2017.

Mamadu, E.J. and Njoseh, I.N., "Tau-collocation approximation approach for solving first and second order ordinary differential equations," Journal of Applied Mathematics and Physics, **4:**383-390. http://www.scrip.org/journal/jamp, 2016.

Ojobor, S.A. and Mamadu, E.J., "Variation iteration decomposition method for the numerical treatment of the Fokker-Plank equation," Nigerian Annals of Natural Sciences, **16(1):**109-113. http://www.nansjournal.org, 2017.

Mamadu, E.J. and Njoseh, I.N., "Solving delay differential equations by Elzaki transform method," Boson Journal of Modern Physics, **3(1):**2454-8413. http://www.scitecresearch.com, 2017.

Njoseh, I.N. and Mamadu, E.J., "Numerical solutions of a generalized Nth order boundary value problems using power series approximation method," Applied Mathematics, **7:**1215-1224. http://www.scirp.org.journal.am, 2016.

Njoseh, I.N. and Mamadu, E.J., "Modified power series approximation method for generalized boundary value problems," Ege Uni. J. of Faculty of Sci, **41(1):**32-43, 2016.

Fox, L. and Parker, I.B., "Chebychev polynomials in numerical analysis". Oxford University Press, Oxford, London, 1968.

Lanczos, C., "Trigonometric interpolation of empirical analytical functions,". J. Math. Phys., **17:**123-199, 1938.

Njoseh, I.N. and Mamadu, E.J. (2016b). Transfomed generate approximation method for generalized boundary value problems using first-kind Chebychev polynomials. Science World Journal, **11(4):**30-33. http://www.scienceworldjournal.org